

# Onset of superradiant instabilities in the composed Kerr-black-hole-mirror bomb

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(Dated: December 22, 2014)

## Abstract

It was first pointed out by Press and Teukolsky that a system composed of a spinning Kerr black hole surrounded by a reflecting mirror may develop instabilities. The physical mechanism responsible for the development of these exponentially growing instabilities is the superradiant amplification of bosonic fields confined between the black hole and the mirror. A remarkable feature of this composed black-hole-mirror-field system is the existence of a critical mirror radius,  $r_m^{\text{stat}}$ , which supports *stationary* (marginally-stable) field configurations. This critical (‘stationary’) mirror radius marks the boundary between stable and unstable black-hole-mirror-field configurations: composed systems whose confining mirror is situated in the region  $r_m < r_m^{\text{stat}}$  are stable (that is, all modes of the confined field decay in time), whereas composed systems whose confining mirror is situated in the region  $r_m > r_m^{\text{stat}}$  are unstable (that is, there are confined field modes which grow exponentially over time). In the present paper we explore this critical (marginally-stable) boundary between stable and explosive black-hole-mirror-field configurations. It is shown that the innermost (*smallest*) radius of the confining mirror which allows the extraction of rotational energy from a spinning Kerr black hole approaches the black-hole horizon radius in the extremal limit of rapidly-rotating black holes. We find, in particular, that this critical mirror radius (which marks the onset of superradiant instabilities in the composed system) scales linearly with the black-hole temperature.

## I. INTRODUCTION

One of the most intriguing phenomenon in black-hole physics is the superradiant scattering of bosonic fields by spinning black holes: it was first pointed out by Zel'dovich [1] that a co-rotating integer-spin wave field of frequency  $\omega$  interacting with a spinning Kerr black hole can be amplified (gain energy) if the composed system is in the superradiant regime [2]

$$\omega < \omega_c \equiv m\Omega_H . \quad (1)$$

Here  $m$  is the azimuthal harmonic index of the incident wave field, and

$$\Omega_H = \frac{a}{r_+^2 + a^2} \quad (2)$$

is the angular velocity of the spinning Kerr black hole [3]. The amplification of the scattered bosonic fields in the superradiant regime (1) is accompanied by a decrease in the rotational energy and angular momentum of the central spinning black hole [4–8].

Soon after Zel'dovich's discovery [1] of the superradiance phenomenon in black-hole physics, it was realized by Press and Teukolsky [9] that the coupled black-hole-field system may develop exponentially growing instabilities. This unstable system, known as the *black-hole bomb*, is composed of three ingredients [9]: (1) a spinning black hole whose rotational energy serves as the energy source of the composed system, (2) a co-rotating bosonic cloud which orbits the central black hole and interacts with it to extract its rotational energy, and (3) a reflecting mirror which surrounds the central black hole and prevents the amplified bosonic field from radiating its energy to infinity [10, 11].

A remarkable feature of this composed physical system is the existence, for each given set  $(\bar{a}, l, m)$  of the black hole and field parameters [12], of a critical (*minimum*) mirror radius which marks the boundary between stable and unstable black-hole-mirror-field configurations. The critical ('stationary' [13]) mirror radius,  $r_m^{\text{stat}}(\bar{a}, l, m)$ , corresponds to *stationary* confined field configurations which are characterized by the critical superradiant frequency (1).

The composed black-hole-mirror-scalar-field system was studied in the important work by Cardoso et. al. [14, 15]. In particular, it was shown in [14] that, for a given set of parameters  $(\bar{a}, l, m)$ , black-hole-mirror-field systems whose mirror radii lie in the regime  $r_m < r_m^{\text{stat}}(\bar{a}, l, m)$  are stable (the confined scalar mode decays in time) whereas black-hole-

mirror-field systems whose mirror radii lie in the regime  $r_m > r_m^{\text{stat}}(\bar{a}, l, m)$  are unstable (the confined scalar mode grows exponentially over time).

The numerical results presented in [14] indicate that the critical (‘stationary’ [13]) mirror radius,  $r_m^{\text{stat}}(\bar{a}, l, m)$ , has the following three important features:

- For a confined field mode of given harmonic indexes  $(l, m)$  [see Eq. (10) below], the stationary mirror radius is a decreasing function of the black-hole angular momentum  $\bar{a}$ . In other words, rapidly-rotating Kerr black holes are characterized by stationary mirror radii which are smaller (closer to the black-hole horizon) than the corresponding stationary mirror radii of slowly-rotating black holes.
- For a given value of the spheroidal harmonic index  $l$  of the confined field, the stationary mirror radius decreases with increasing values of the azimuthal harmonic index  $m$ . The equatorial  $m = l$  mode is therefore characterized by the innermost (smallest) stationary mirror radius among confined field modes which share the same spheroidal harmonic index  $l$ .
- For confined equatorial  $m = l$  modes, the stationary mirror radius decreases with increasing values of the spheroidal harmonic index  $l$ .

From these characteristics of the stationary mirror radius, one concludes that the asymptotic radius

$$r_m^* \equiv r_m^{\text{stat}}(\bar{a} \rightarrow 1, l = m \rightarrow \infty) \quad (3)$$

provides the innermost location of the confining mirror. The physical significance of this critical mirror radius,  $r_m^*$ , lies in the fact that this is the innermost (*smallest*) radius of the confining mirror which allows the extraction of rotational energy from spinning Kerr black holes. Below we shall explore the physical properties of this critical mirror radius.

Cardoso et. al. [14] also provided an analytic treatment of the black-hole-mirror-field system in the *small* frequency regime  $a\omega \ll 1$ . Substituting the critical (marginal) super-radiant frequency  $\omega_c = m\Omega_H$  [see Eq. (1)] into equation (30) of [14], one finds that the stationary mirror radius  $r_m^{\text{stat}}(\bar{a}, l, m)$  is given as a solution of the characteristic equation

$$J_{l+1/2}(m\Omega_H r_m^{\text{stat}}) = 0, \quad (4)$$

where  $J_{l+1/2}(x)$  is the Bessel function. It should be emphasized that the characteristic equation (4) for the ‘stationary’ (critical) radii of the mirror is valid in the regime  $a\omega \ll 1$

considered in [14]. Thus, the characteristic equation (4) can only determine the stationary mirror radii of *slowly*-rotating black holes in the regime  $m\bar{a} \ll 1$ .

One of the goals of the present study is to obtain an analogous characteristic equation for the stationary mirror radii of *rapidly*-rotating ( $\bar{a} \simeq 1$ ) black holes. As discussed above, the *numerical* results presented in [14] indicate that these near-extremal black holes are characterized by stationary mirror radii which are *closer* to the black-hole horizon than the corresponding stationary mirror radii (4) of slowly-rotating black holes. Below we shall confirm this expectation *analytically*.

In addition, as we shall show below, our characteristic equation for the critical (‘stationary’ [13]) mirror radii of rapidly-rotating black holes [see Eq. (21) below] is valid for arbitrarily large values of the harmonic indexes  $(l, m)$  of the confined field [16]. This fact will allow us to address the following interesting question regarding the nature of this critical mirror radius: what is the *asymptotic* behavior of the stationary mirror radius in the eikonal  $l \gg 1$  limit?

As discussed above, the asymptotic mirror radius,  $r_m^* \equiv r_m^{\text{stat}}(\bar{a} \rightarrow 1, l = m \rightarrow \infty)$ , corresponds to the *innermost* location of the confining mirror which allows the extraction of rotational energy from spinning Kerr black holes. In addition, for generic confined field configurations [17], this critical mirror radius marks the boundary between stable and explosive black-hole-mirror-field configurations. One of the goals of the present study is to determine this fundamental (asymptotic) mirror radius  $r_m^*$ .

## II. DESCRIPTION OF THE SYSTEM

The explored physical system is composed of a spinning Kerr black hole of mass  $M$  and angular momentum  $Ma$  linearly coupled to a massless scalar field  $\Psi$ . In the Boyer-Lindquist coordinate system  $(t, r, \theta, \phi)$  the black-hole spacetime geometry is described by the line-element [18, 19]

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar \sin^2 \theta}{\rho^2} dt d\phi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2 r \sin^2 \theta}{\rho^2}\right) \sin^2 \theta d\phi^2 \quad (5)$$

where  $\Delta \equiv r^2 - 2Mr + a^2$  and  $\rho^2 \equiv r^2 + a^2 \cos^2 \theta$ . The zeroes of  $\Delta$  determine the radii of the black-hole (event and inner) horizons:

$$r_{\pm} = M \pm (M^2 - a^2)^{1/2}. \quad (6)$$

As discussed above, the critical (‘stationary’ [13]) mirror radius which characterizes the composed black-hole-mirror-field system is a decreasing function of the black-hole rotation parameter  $\bar{a}$ . Thus, rapidly-rotating black holes are expected to be characterized by the smallest (*innermost*) stationary mirror radii. In the present study we shall analyze the physical properties of the black-hole-mirror-field system in this physically interesting regime of rapidly-rotating Kerr black holes with

$$\bar{a} \simeq 1 . \quad (7)$$

The dynamics of the scalar field  $\Psi$  in the curved geometry is determined by the Klein-Gordon wave equation

$$\nabla^a \nabla_a \Psi = 0 , \quad (8)$$

which, in the rotating Kerr spacetime (5) becomes [20, 21]

$$\left[ \frac{(r^2 + a^2)^2}{\Delta} - a^2 \sin^2 \theta \right] \frac{\partial^2 \Psi}{\partial t^2} + \frac{4Mar}{\Delta} \frac{\partial^2 \Psi}{\partial t \partial \phi} + \left( \frac{a^2}{\Delta} - \frac{1}{\sin^2 \theta} \right) \frac{\partial^2 \Psi}{\partial \phi^2} - \Delta \frac{\partial}{\partial r} \left( \Delta \frac{\partial \Psi}{\partial r} \right) - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Psi}{\partial \theta} \right) = 0 . \quad (9)$$

It is convenient to decompose the scalar field  $\Psi$  in the form

$$\Psi = \sum_{l,m} e^{im\phi} S_{lm}(\theta; a\omega) R_{lm}(r; a, \omega) e^{-i\omega t} , \quad (10)$$

in which case one finds [20] that the radial  $R_{lm}$  and angular  $S_{lm}$  wave functions are determined by two coupled ordinary differential equations of the confluent Heun type [22–25], see Eqs. (11) and (13) below.

It is worth emphasizing that the sign of  $\Im\omega$  in (10) reflects the stability/instability properties of the confined field mode: stable modes (modes which decay exponentially in time) are characterized by  $\Im\omega < 0$ , whereas unstable modes (modes which grow exponentially over time) are characterized by  $\Im\omega > 0$ . Stationary (marginally-stable) field modes are characterized by  $\Im\omega = 0$ .

The angular functions  $S_{lm}(\theta; a\omega)$  are known as the spheroidal harmonic functions. These functions are solutions of the angular differential equation [22, 23, 25, 26]

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dS_{lm}}{d\theta} \right) + \left( a^2 \omega^2 \cos^2 \theta - \frac{m^2}{\sin^2 \theta} + A_{lm} \right) S_{lm} = 0 \quad (11)$$

in the interval  $\theta \in [0, \pi]$ . Regular eigenfunctions [27] exist for a discrete set  $\{A_{lm}(a\omega)\}$  of angular eigenvalues which are labeled by the two integers  $m$  and  $l \geq |m|$ . These angular eigenvalues can be expanded in the form [28–30]

$$A_{lm}(a\omega) = l(l+1) + \sum_{k=1}^{\infty} c_k(a\omega)^k, \quad (12)$$

where the expansion coefficients  $\{c_k(l, m)\}$  are given in [28–30].

The radial functions  $R_{lm}(r; a, \omega)$  satisfy the ordinary differential equation [22]

$$\Delta \frac{d}{dr} \left( \Delta \frac{dR_{lm}}{dr} \right) + \left[ K^2 - \Delta(a^2\omega^2 - 2ma\omega + A_{lm}) \right] R_{lm} = 0, \quad (13)$$

where  $K \equiv (r^2 + a^2)\omega - ma$ . Note that the angular differential equation (11) and radial differential equation (13) are coupled by the angular eigenvalues  $\{A_{lm}(a\omega)\}$ . (We shall henceforth omit the harmonic indexes  $l$  and  $m$  for brevity.)

We shall be interested in solutions of the radial wave equation (13) with the physical requirement (boundary condition) of purely ingoing waves (as measured by a comoving observer) crossing the black-hole horizon [22]. As shown in [22], this boundary condition corresponds to the behavior

$$R \sim e^{-i(\omega - m\Omega_H)y} \quad \text{as } r \rightarrow r_+ \quad (y \rightarrow -\infty) \quad (14)$$

of the radial eigenfunction in the vicinity of the black-hole horizon, where the “tortoise” radial coordinate  $y$  is defined by  $dy = [(r^2 + a^2)/\Delta]dr$ . In addition, following [7, 14] we shall assume that the scalar field vanishes at the location  $r_m$  of the confining mirror:

$$R(r = r_m) = 0. \quad (15)$$

For the analysis of the radial Teukolsky equation (13), it is convenient to define new dimensionless variables

$$x \equiv \frac{r - r_+}{r_+} \quad ; \quad \tau \equiv 8\pi MT_{\text{BH}} = \frac{r_+ - r_-}{r_+} \quad ; \quad k = 2\omega r_+ \quad ; \quad \varpi \equiv \frac{\omega - m\Omega_H}{2\pi T_{\text{BH}}}, \quad (16)$$

in terms of which Eq. (13) becomes

$$x(x + \tau) \frac{d^2 R}{dx^2} + (2x + \tau) \frac{dR}{dx} + VR = 0, \quad (17)$$

where  $V \equiv K^2/r_+^2 x(x + \tau) - (a^2\omega^2 - 2ma\omega + A)$  and  $K = r_+^2\omega^2 + r_+kx + r_+\varpi\tau/2$ .

### III. MARGINALLY STABLE BLACK-HOLE-MIRROR-FIELD CONFIGURATIONS

We shall now analyze the *stationary* ( $\Im\omega = 0$ ) resonances of the composed system. As discussed above, these stationary (marginally-stable) black-hole-mirror-field configurations are characterized by the critical frequency (1) for superradiant scattering in the black-hole spacetime. In particular, in this section we shall find the *discrete* set of mirror radii,  $\{r_m^{\text{stat}}(\bar{a}, l, m; n)\}$ , which support stationary confined field configurations. (Here  $n = 1, 2, 3, \dots$  is the resonance parameter).

We consider a rapidly-rotating Kerr black hole [31] surrounded by a reflecting mirror which is placed in the vicinity of the black-hole horizon. In particular, we shall assume the following inequalities:

$$\tau \ll 1 \quad \text{and} \quad x_m \ll 1 . \quad (18)$$

In the near-horizon region  $x \ll 1$  the radial equation is given by (17) with  $V \rightarrow V_{\text{near}} \equiv -(a^2\omega^2 - 2ma\omega + A) + (kx + \varpi\tau/2)^2/x(x + \tau)$ . The physical radial solution obeying the ingoing boundary condition (14) at the black-hole horizon with the critical (marginally-stable) superradiant frequency  $\varpi = 0$  is given by [22, 30]

$$R(x) = \left(\frac{x}{\tau} + 1\right)^{-ik} {}_2F_1\left(\frac{1}{2} - ik + i\delta, \frac{1}{2} - ik - i\delta; 1; -x/\tau\right) , \quad (19)$$

where  ${}_2F_1(a, b; c; z)$  is the hypergeometric function [30], and

$$\delta^2 \equiv -a^2\omega^2 + 2ma\omega - A + k^2 - \frac{1}{4} . \quad (20)$$

We shall henceforth consider the case of real  $\delta$  [32, 33]. The mirror-like boundary condition  $R(x = x_m^{\text{stat}}) = 0$  [see Eq. (15)] now reads

$${}_2F_1(1/2 - ik + i\delta, 1/2 - ik - i\delta; 1; -x_m^{\text{stat}}/\tau) = 0 . \quad (21)$$

It is worth emphasizing that, the newly derived resonance condition (21) for the critical (‘stationary’ [13]) mirror radii of the system is valid for confining mirrors which are placed in the near-horizon region  $x_m \ll 1$ . [Below we shall see that this near-horizon condition implies that the stationary mirror radii obtained from (21) are valid in the regime of *rapidly*-rotating black holes with  $\tau \ll 1$ ]. On the other hand, the resonance condition (4) [14] for

the stationary mirror radii is valid in the complementary regime  $x_m \gg 1$ , or equivalently in the regime of *slowly*-rotating black holes with  $m\bar{a} \ll 1$ .

One important conclusion which can immediately be drawn from the resonance condition (21) is the fact that, for rapidly-rotating black holes, the critical mirror radius scales linearly with the black-hole temperature [34]:

$$x_m^{\text{stat}} \propto \tau . \quad (22)$$

This implies that the stationary mirror radius  $x_m^{\text{stat}}$  is a decreasing function of the black-hole rotation parameter  $\bar{a}$  (an increasing function of the black-hole dimensionless temperature  $\tau$ ).

As we shall now show, for small and moderate values of the field harmonic indexes  $(l, m)$ , the stationary mirror radii  $x_m^{\text{stat}}(\bar{a}, l, m; n)$  are characterized by the relation

$$x_m^{\text{stat}}/\tau \gg 1 , \quad (23)$$

in which case the resonance condition (21) can be solved *analytically*. In the regime (23) one can use the large- $z$  asymptotic behavior of the hypergeometric function  ${}_2F_1(a, b; c; z)$  [30] to approximate the resonance condition (21) by

$$\frac{\Gamma(2i\delta)}{\Gamma(1/2 - ik + i\delta)\Gamma(1/2 + ik + i\delta)} \left(\frac{x_m^{\text{stat}}}{\tau}\right)^{i\delta} + \frac{\Gamma(-2i\delta)}{\Gamma(1/2 - ik - i\delta)\Gamma(1/2 + ik - i\delta)} \left(\frac{x_m^{\text{stat}}}{\tau}\right)^{-i\delta} = 0 , \quad (24)$$

which yields

$$\left(\frac{x_m^{\text{stat}}}{\tau}\right)^{2i\delta} = -\frac{\Gamma(-2i\delta)\Gamma(1/2 - ik + i\delta)\Gamma(1/2 + ik + i\delta)}{\Gamma(2i\delta)\Gamma(1/2 - ik - i\delta)\Gamma(1/2 + ik - i\delta)} . \quad (25)$$

It is easy to verify that, for real values of the angular eigenvalue  $\delta$ , the characteristic equation (25) corresponds to real values of the stationary mirror radii [15]: taking the logarithm of both sides of (25), one obtains the characteristic equation [35]

$$2\delta \ln \left(\frac{x_m^{\text{stat}}}{\tau}\right) = i \ln \left[\frac{\Gamma(2i\delta)}{\Gamma(-2i\delta)}\right] + i \ln \left[\frac{\Gamma(1/2 - ik - i\delta)}{\Gamma(1/2 + ik + i\delta)}\right] + i \ln \left[\frac{\Gamma(1/2 + ik - i\delta)}{\Gamma(1/2 - ik + i\delta)}\right] + \pi(2n-1) \quad (26)$$

for the discrete set  $\{x_m^{\text{stat}}(\bar{a}, l, m; n)\}$  of stationary mirror radii, where  $n = 1, 2, 3, \dots$  is the resonance parameter of the mode. Inspection of the resonance condition (26) reveals the following facts: the first three terms on the r.h.s of this equation are of the form  $i[\ln(z) - \ln(\bar{z})]$  and are therefore purely real numbers (see Eq. 6.1.23 of [30]). The fourth term on the r.h.s of (26) is obviously a purely real number. This implies that the r.h.s of (26) is a purely



real number. One therefore concludes that, for real values of the angular eigenvalue  $\delta$ , the dimensionless mirror radii obtained from the resonance condition (26) are purely real numbers. These stationary [13] mirror radii are given by

$$x_{\text{m}}^{\text{stat}}(n) = \tau \times \left[ \frac{\Gamma(-2i\delta)\Gamma(1/2 + ik + i\delta)\Gamma(1/2 - ik + i\delta)}{\Gamma(2i\delta)\Gamma(1/2 - ik - i\delta)\Gamma(1/2 + ik - i\delta)} \right]^{1/2i\delta} e^{\pi(2n-1)/2\delta}. \quad (27)$$

In Table I we display the discrete radii of the reflecting mirror corresponding to the stationary confined equatorial  $l = m = 2$  mode as obtained from the analytical formula (27) [36]. We also display the corresponding mirror radii as obtained from a direct numerical solution [37] of the characteristic resonance equation (21). One finds a remarkably good agreement between the exact mirror radii [as obtained numerically from the resonance condition (21)] and the approximated mirror radii [as obtained from the analytical formula (27)].

Formula	$x_{\text{m}}^{\text{stat}}(n = 1)/\tau$	$x_{\text{m}}^{\text{stat}}(n = 2)/\tau$	$x_{\text{m}}^{\text{stat}}(n = 3)/\tau$	$x_{\text{m}}^{\text{stat}}(n = 4)/\tau$
Analytical [Eq. (27)]	94.06	2613.20	72603.50	2017168.1
Numerical [Eq. (21)]	91.79	2610.95	72601.28	2017165.8

TABLE I: Stationary resonances of the composed black-hole-mirror-field system. We display the scaled mirror radii,  $x_{\text{m}}^{\text{stat}}(l = m = 2; n)/\tau$ , corresponding to the stationary confined mode with  $l = m = 2$ , as obtained from a direct numerical solution of the characteristic resonance equation (21). We also display the corresponding mirror radii as obtained from the analytical formula (27). One finds a remarkably good agreement between the exact (numerically computed) mirror radii and the approximated (analytically calculated) mirror radii.

In Table II we display the discrete set of mirror radii,  $\{x_{\text{m}}^{\text{stat}}(l = m; n)/\tau\}$ , corresponding to composed black-hole-mirror configurations with stationary confined equatorial ( $l = m$ ) field modes [38–40]. The data presented in Table II correspond to a direct numerical solution [37] of the characteristic resonance condition (21). It is worth recalling that, for a confined pure field (that is, a confined mode characterized by a given value of the azimuthal harmonic index  $m$ ), the smallest stationary mirror radius,  $x_{\text{m}}^{\text{stat}}(\bar{a}, m; n = 1)$ , marks the boundary between stable and unstable black-hole-mirror-field configurations for that particular pure mode.

From Table II one learns that the stationary mirror radius,  $x_{\text{m}}^{\text{stat}}(\bar{a}, m; n = 1)$ , is a *decreasing* function of the azimuthal harmonic index  $m$ . This is a generic feature of the composed

black-hole-mirror-field system: in Table III we display the values of the stationary mirror radii in the asymptotic  $m \gg 1$  regime [41, 42]. One finds that the data presented in Table III is described extremely well by the simple asymptotic formula:

$$\frac{x_m^{\text{stat}}(\bar{a} \rightarrow 1, l = m \gg 1)}{\tau} \simeq \alpha + \frac{\beta}{m} + O(m^{-2}) \quad \text{with} \quad \alpha \simeq 0.36 \quad ; \quad \beta \simeq 10.5 . \quad (28)$$

What we find most interesting is the fact that the coefficient  $\alpha$  in (28) has a finite asymptotic value. This fact suggests that the composed Kerr-black-hole-mirror-field system is characterized by a *finite* asymptotic limit of the critical (‘stationary’) mirror radius:

$$\frac{x_m^*}{\tau} \simeq 0.36 , \quad (29)$$

where  $x_m^* \equiv x_m^{\text{stat}}(\bar{a} \rightarrow 1, l = m \rightarrow \infty)$  [see Eq. (3)].

This feature of the *spinning* black-hole-mirror-field bomb should be contrasted with the corresponding case of the *charged* black-hole-mirror-field bomb [6, 7]. In the charged case one finds [7] that the critical (‘stationary’) mirror radius  $x_m^{\text{stat}}$  can be made arbitrarily small (that is, the mirror can be placed arbitrarily close to the black-hole horizon) in the asymptotic [43]  $qQ \gg 1$  regime [7]:

$$\frac{x_m^{\text{stat}}(qQ \gg 1)}{\tau} = O(1/qQ) \rightarrow 0 . \quad (30)$$

$l = m$	$\delta$	$x_m^{\text{stat}}(n = 1)/\tau$	$x_m^{\text{stat}}(n = 2)/\tau$	$x_m^{\text{stat}}(n = 3)/\tau$	$x_m^{\text{stat}}(n = 4)/\tau$
2	0.945	91.79	2610.95	72601.28	2017165.8
3	1.937	10.99	62.43	322.69	1640.27
4	2.849	5.23	18.85	59.77	183.00
5	3.739	3.45	9.95	24.93	59.59

TABLE II: Stationary resonances of the composed black-hole-mirror-field system. We display the dimensionless mirror radii,  $x_m^{\text{stat}}(m; n)/\tau$ , corresponding to stationary confined modes. The data presented is for the equatorial  $l = m$  modes, the modes with the smallest (innermost) stationary mirror radii. Also shown are the corresponding values of the angular eigenvalues  $\delta$  [see Eq. (20)]. One finds that the critical (stationary) radius of the mirror,  $x_m^{\text{stat}}(m; n)$ , is a decreasing function of the azimuthal harmonic index  $m$ .

In order to support our findings, according to which the composed Kerr-black-hole-mirror-field system is characterized by a *finite* asymptotic value of the critical mirror radius, we

$l = m$	25	50	75	100	125	150	175	200
$x_m^{\text{stat}}(m; n = 1)/\tau$	0.75	0.55	0.49	0.46	0.44	0.43	0.42	0.41

TABLE III: Stationary resonances of the composed black-hole-mirror-field system. We display the dimensionless mirror radii,  $x_m^{\text{stat}}(m; n = 1)/\tau$ , corresponding to stationary confined modes in the asymptotic  $l = m \gg 1$  regime. One finds that the critical (stationary) radius of the mirror,  $x_m^{\text{stat}}(m; n = 1)$ , decreases monotonically to an asymptotic *finite* value [see Eq. (29)] in the eikonal  $l \gg 1$  limit.

shall prove in the next section that composed systems whose mirror radii lie in the regime  $x_m/\tau \ll 1$  *cannot* support stationary confined field configurations.

#### IV. NO STATIONARY CONFINED FIELD CONFIGURATIONS IN THE REGIME $x_m/\tau \ll 1$

We have seen that the (scaled) critical mirror radius,  $x_m^{\text{stat}}(m)/\tau$ , decreases monotonically with increasing values of the azimuthal harmonic index  $m$ . This fact naturally gives rise to the following question: Can the ratio  $x_m^{\text{stat}}(m)/\tau$  be made arbitrarily small in the asymptotic eikonal regime  $m \rightarrow \infty$ ? One can also formulate this question in a more practical form: Is it possible to extract the black-hole rotational energy by placing a confining mirror arbitrarily close to its horizon [44]?

Our analysis in Sec. III provides compelling evidence that the answer to the above questions is “no”. In particular, our results suggest that the Kerr-black-hole-mirror-field system is characterized by a *finite* asymptotic value of the critical (‘stationary’ [13]) mirror radius:  $x_m^{\text{stat}}(m \gg 1)/\tau = O(1)$  [see Eq. (29)]. In order to support this finding, we shall prove in this section that composed black-hole-mirror systems whose mirror radii lie in the regime  $x_m/\tau \ll 1$  cannot support stationary confined field configurations [45].

To that end, it proves useful to write the radial Teukolsky equation in the form of a Schrödinger-like wave equation [46]:

$$\frac{d^2\psi}{dy^2} + V\psi = 0, \quad (31)$$

where  $\psi = (r^2 + a^2)^{1/2}R$  and the “tortoise” radial coordinate  $y$  was defined in Sec. II. The

potential  $V$  in Eq. (31) is given by [46]

$$V = \frac{K^2 - \Delta\lambda}{(r^2 + a^2)^2} - G^2 - \frac{dG}{dy} , \quad (32)$$

where

$$G = \frac{r\Delta}{(r^2 + a^2)^2} , \quad (33)$$

and

$$\lambda = k^2 - \frac{1}{4} - \delta^2 . \quad (34)$$

In the near-horizon region [47]

$$x \ll \tau , \quad (35)$$

one finds [48]

$$y \simeq \frac{r_+^2 + a^2}{r_+ - r_-} \ln \left( \frac{r - r_+}{r_+ - r_-} \right) = \frac{2M}{\tau} \ln \left( \frac{x}{\tau} \right) , \quad (36)$$

which implies  $\Delta \simeq (r_+ - r_-)^2 e^{\tau y/2M}$  and

$$V(y) \simeq (\omega - m\Omega_H)^2 - V_H e^{\tau y/2M} , \quad (37)$$

where

$$V_H \equiv \left( \frac{\tau}{2M} \right)^2 \left( \lambda + \frac{\tau r_+}{2M} \right) . \quad (38)$$

In the eikonal regime,  $l, m \gg 1$ , one can use the relation [42]

$$\delta = l \times \sqrt{-1 + \frac{15}{8}\mu^2 - \frac{1}{8}\mu^4} + O(1) \quad ; \quad \mu \equiv \frac{m}{l} , \quad (39)$$

in order to find [see Eq. (34)]

$$\lambda = l^2 \left( 1 - \frac{7}{8}\mu^2 + \frac{1}{8}\mu^4 \right) . \quad (40)$$

From (40) one concludes that

$$\lambda > 0 \quad (41)$$

for all values of the dimensionless ratio  $\mu$  (note that  $\mu \leq 1$ ).

Taking cognizance of Eqs. (31), (37), and (38), one obtains the radial wave equation

$$\frac{d^2\psi}{dy^2} - V_H e^{\tau y/2M} \psi = 0 \quad (42)$$

for stationary field configurations (with  $\omega = m\Omega_H$ ). Defining

$$z = \frac{\tau}{4M} y , \quad (43)$$

the radial wave equation (42) can be written in the form

$$\frac{d^2\psi}{dz^2} - 4\left(\lambda + \frac{\tau r_+}{2M}\right)e^{2z}\psi = 0. \quad (44)$$

Using Eq. (9.1.54) of [30], one finds that the physical solution to the radial equation (44) is described by the Bessel function of the first kind:

$$\psi(z) = J_0\left(2i\sqrt{\lambda + \frac{\tau r_+}{2M}}e^z\right). \quad (45)$$

This radial solution can also be written [see Eqs. (36) and (43)] in the form [49]

$$\psi(x) = J_0\left(2i\sqrt{\left(\lambda + \frac{\tau r_+}{2M}\right)\frac{x}{\tau}}\right). \quad (46)$$

Taking cognizance of Eq. (41), one finds that the argument of the Bessel function in (46) is purely *imaginary*. It is well known that the Bessel function  $J_0(ix)$  with  $x \in \mathbb{R}$  has no zeroes [that is,  $\psi(x_m) \neq 0$  for  $x_m \in \mathbb{R}$ ]. One therefore concludes that stationary solutions (with  $\omega = m\Omega_H$ ) of the wave field are *not* compatible with the mirror-like boundary condition (15) for confining mirrors in the region  $x_m \ll \tau$  [see Eq. (35)].

The proof presented in this section supports our previous conclusion that, the composed Kerr-black-hole-mirror-field system is characterized by a *finite* asymptotic value [see Eq. (29)] of the dimensionless ratio  $x_m^*/\tau$ , where  $x_m^*$  is the critical mirror radius.

## V. SUMMARY AND DISCUSSION

In this paper, we have used analytical tools in order to study the *stationary* (marginally-stable) resonances of the composed black-hole-mirror-field system. These resonances are fundamental to the physics of confined bosonic fields in black-hole spacetimes. In particular, these resonances mark the onset of superradiant instabilities in the black-hole bomb mechanism of Press and Teukolsky [9].

We have derived the characteristic resonance condition (21) for the marginally-stable (stationary) black-hole-mirror-field configurations. In particular, it was shown that, for rapidly-rotating black holes, the stationary resonances of the system are described by the simple zeroes of the hypergeometric function.

The characteristic resonance condition (21) determines the discrete set of mirror radii,  $\{x_m^{\text{stat}}(\bar{a}, l, m; n)\}$ , which support stationary confined field configurations in the black-hole

spacetime. One nice feature of this resonance condition lies in the fact that it immediately reveals that the critical (‘stationary’ [13]) mirror radii scale linearly with the black-hole temperature [34], see Eq. (22). This fact implies that  $x_m^{\text{stat}}(\bar{a}, l, m; n)$  is a decreasing function of the black-hole rotation parameter  $\bar{a}$  — the larger the black-hole spin, the closer to the black-hole horizon the confining mirror can be placed.

It was shown that the stationary mirror radius,  $x_m^{\text{stat}}(m)$ , decreases monotonically with increasing values of the azimuthal harmonic index  $m$  of the confined field mode. In particular, our results provide compelling evidence that the composed Kerr-black-hole-mirror-field system is characterized by a *finite* asymptotic value of the critical mirror radius:  $x_m^{\text{stat}}(m \gg 1)/\tau \simeq 0.36$  [50].

The physical significance of the asymptotic stationary mirror radius,  $x_m^* \equiv x_m^{\text{stat}}(\bar{a} \rightarrow 1, l = m \rightarrow \infty)$ , lies in the fact that it is the *innermost* location of the confining mirror which allows the extraction of rotational energy from spinning black holes. This implies that, for generic confined field configurations [17], this critical mirror radius marks the onset of superradiant instabilities in the composed system: composed black-hole-mirror-field systems whose mirror radii lie in the regime  $x_m < x_m^*$  are stable (that is, all modes of the confined field decay in time), whereas composed black-hole-mirror-field systems whose mirror radii lie in the regime  $x_m > x_m^*$  are unstable (that is, there are confined field modes which grow exponentially over time).

## ACKNOWLEDGMENTS

This research is supported by the Carmel Science Foundation. I thank Yael Oren, Arbel M. Ongo and Ayelet B. Lata for helpful discussions.

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  - [2] We use natural units in which  $G = c = \hbar = 1$ .
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- [12] Here  $\bar{a} \equiv J/M^2$  is the dimensionless angular momentum of the black hole and  $(l, m)$  are respectively the spheroidal harmonic index and the azimuthal harmonic index of the confined field mode, see Eq. (10) below.
- [13] We use the terminology ‘stationary mirror radius’ in order to reflect the fact that this critical radius of the confining mirror supports *stationary* black-hole-field configurations with  $\Im\omega = 0$ . It is worth emphasizing again that, for a confined field mode of given harmonic indexes  $(l, m)$ , the stationary mirror radius  $r_m^{\text{stat}}(\bar{a}, l, m)$  marks the boundary between stable and unstable black-hole-field configurations. Thus, this critical mirror radius signals the onset of the superradiant instabilities in the composed black-hole-mirror-field system.
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- [32] Below we shall see that equatorial field modes with  $l = m \geq 2$ , the modes we shall be interested in in this study, are characterized by this property.
- [33] One can choose  $\delta > 0$  without loss of generality.
- [34] Note that the resonance condition (21) for the stationary confined states is expressed in terms of the dimensionless ratio  $x_m^{\text{stat}}/\tau$ .
- [35] Here we have used the relation  $-1 = e^{i\pi(2n-1)}$  which implies  $\ln(-1) = i\pi(2n-1)$ , where  $n$  is an integer.
- [36] We find from (27) that modes with  $n \leq 0$  are characterized by  $x_m^{\text{stat}}/\tau < 10$ . These modes violate the assumption (23) and thus should not be regarded as physical ones. [It is worth emphasizing again that the analytical formula (27) for the critical (stationary) mirror radii is valid in the regime  $\tau \ll x_m^{\text{stat}} \ll 1$ , see Eqs. (18) and (23)].
- [37] It is worth emphasizing that, finding the roots of the characteristic equation (21) is a much easier task than the Runge-Kutta numerical solution of the Teukolsky wave equation pre-

sented in [14]. In addition, as we shall discuss below, another notable advantageous of our analysis lies in the fact that, the characteristic resonance condition (21) can easily be solved for asymptotically *large* values of the harmonic indexes  $(l, m)$ . Below we shall be interested in this asymptotic (eikonal) regime.

- [38] One of our goals is to determine the *smallest* mirror radius  $r_m^*$  [see Eq. (3)] which allows the extraction of rotational energy from spinning Kerr black holes. (This critical mirror radius also marks the boundary between stable and unstable black-hole-mirror-field configurations.) As discussed above, the numerical results of [14] indicate that, for a given value of the spheroidal harmonic index  $l$  of the confined field, the equatorial  $m = l$  mode is characterized by the *innermost* location of the stationary mirror radius. We therefore focus our attention on these equatorial modes.
- [39] It is worth emphasizing again that the mirror-like boundary condition (21) is valid for mirror radii in the  $x_m \ll 1$  regime [see Eq. (18)]. Solving the resonance condition (21) for small and moderate values of the spheroidal harmonic index  $l$ , one finds  $x_m^{\text{stat}}(\bar{a}, m; n = 1)/\tau \gtrsim 1$ , see Table II. Thus, the stationary mirror radii presented in Table II are valid in the regime of rapidly-rotating (near-extremal) black holes with  $\tau \lesssim x_m^{\text{stat}}(\bar{a}, m; n = 1) \ll 1$ .
- [40] We are interested here in the relation  $x_m^{\text{stat}} = x_m^{\text{stat}}(\tau)$  to leading-order in the small parameter  $\tau \ll 1$  (note that  $\tau \ll 1$  corresponds to  $\bar{a} \simeq 1$ ). We therefore take  $a\omega = m/2 + O(\tau)$  and  $k = m + O(\tau^2)$  in Eqs. (12), (16), and (20). The leading-order values of the separation constants (the angular eigenvalues)  $\delta_{lm}$  are given in Table II.
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- [44] The answer to this question may one day be of practical importance: In order to save construction materials, one would like to build the confining mirror as close as possible to the black-hole horizon.
- [45] The hypergeometric function  ${}_2F_1(a, b; c; z)$  is characterized by the property  ${}_2F_1(a, b; c; z) \rightarrow 1$  for  $a \cdot b \cdot z/c \ll 1$  [30]. Taking cognizance of the characteristic resonance condition (21), one realizes that there are no confined stationary field configurations in the regime  $m^2 x_m/\tau \ll 1$

[In this regime one finds  ${}_2F_1 \simeq 1$ , whereas the resonance condition (21) for confined stationary states requires  ${}_2F_1 \simeq 0$ ]. In this section we would like to prove the (*stronger*) claim that there are no stationary confined field configurations for mirror radii in the regime  $x_m/\tau \ll 1$ .

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[48] Note that  $y \rightarrow -\infty$  (and thus  $e^{\tau y/M} \rightarrow 0$ ) in the near-horizon region (35).

[49] We recall that our analysis in this section is valid in the regime  $x/\tau \ll 1$  [see Eq. (35)]. However, since  $\lambda = O(l^2)$  [see Eq. (40)], the argument  $\sqrt{(\lambda + \frac{\tau r_+}{2M})\frac{x}{\tau}}$  of the Bessel function in (46) can be large in the asymptotic eikonal  $l \gg 1$  regime.

[50] It is worth emphasizing again that this feature of the *spinning* black-hole-mirror-field bomb is a non-trivial one. In particular, it should be contrasted with the *charged* black-hole-mirror-field bomb [6, 7]. It was previously shown [7] that, in the charged case, the stationary mirror radius,  $x_m^{\text{stat}}(qQ)$  [43], can be made arbitrarily small in the asymptotic  $qQ \gg 1$  regime (that is, in the charged case, the confining mirror can be placed arbitrarily close to the black-hole horizon [7]).